Sydney Girls High School

November 2014

MATHEMATICS
EXTENSION 1

YEAR 12
ASSESSMENT TASK 1 for HSC 2015

Time Allowed: 60 minutes
+ 5 minutes Reading Time

Topics: Mathematical Induction, Parametric Equations and Circle Geometry.

General Instructions:

- There are Seven (7) Questions of equal marks.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Question on a new page.
- Write on one side of paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your student number clearly at the top of each question and clearly number each question.

Total: 56 marks

Student Name:___________________________ **Teacher Name:**____________________

Question 1 (8 marks)

a) Find the equation of the tangent to the parabola $x = 6t$, $y = 3t^2$ at the point where $t = 2$. 3

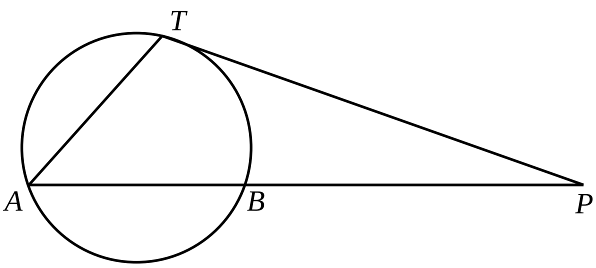
b) Normals are drawn at two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ to the parabola $x^2 = 4ay$.

i) Derive the equation of the normal at P. 2

ii) Find the point of intersection of these two normals. 3

Question 2 (8 marks)

a) Copy or trace the diagram onto your answer page.



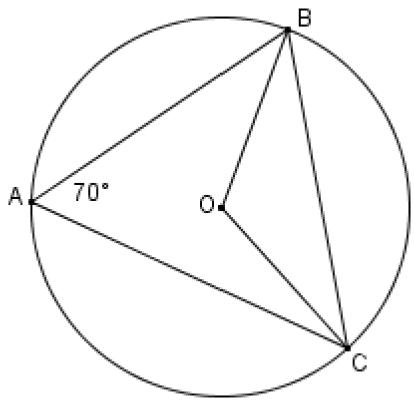
i) PT is a tangent to the circle. Prove that $\Delta TBP \parallel\!| \Delta ATP$. 2

ii) Given that TP = 12 cm and AP = 16 cm, find BP. 2

b) Prove by mathematical induction that $3^{4n} - 1$ is divisible by 80 for $n \geq 1$. 4

Question 3 (8 marks)

a)



i) Find $\angle COB$ with reasons. 2

ii) Find $\angle OCB$ with reasons. 2

b) Prove by Mathematical induction that $4^n \geq 1+3n$ for all $n \geq 1$. 4

Question 4 (8 marks)

a) Derive the Cartesian equation for $x = 2t-1, y = 3t^2+1$ 2

b) The chord PQ of the parabola $x^2 = 12y$ passes through the point $C(8,0)$.
If $P(6p, 3p^2)$ and $Q(6q, 3q^2)$.

i) Derive the equation of the chord PQ. 2

ii) Show that $4(p+q) = 3pq$. 1

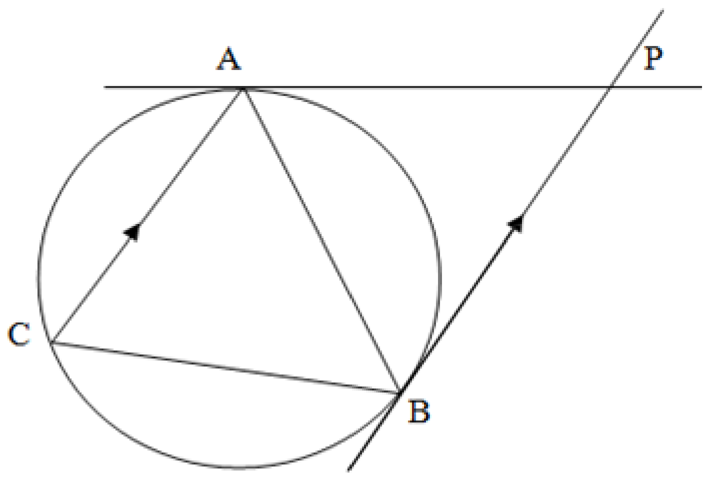
iii) Find the locus of M, midpoint of PQ. 3

Question 5 (8 marks)

a) Consider the points A, B and C lying on the circle, with tangents drawn from A and B, meeting at P. AC is parallel to BP. Copy the diagram.

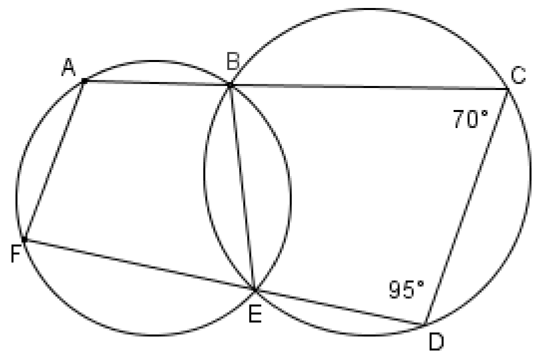
i) Let $\angle CAB = x^\circ$. Prove $AB = BC$. 2

ii) Prove $\angle ABC = \angle APB$. 2



b) $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$.
The chord PQ subtends a right angle at the origin O. Show that $pq = -4$. 2

c) Find $\angle AFE$, give reasons. 2



Question 6 (8 marks)

a) Use Mathematical Induction to prove for all integers $n \ge 1$ 4

$1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = (n-1)2^{n+1} + 2$

b) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The tangent at *P* and the line through *Q* parallel to the y axis intersect at *T*.

i) Draw a diagram to illustrate the information above. 1

ii) Show that the coordinates of *T* are $(2aq, 2apq - ap^2)$, given the tangent at *P* is $y = px - ap^2$ 1

iii) Find the locus of M, the midpoint of PT when $pq = -1$. 2

Question 7 (8 marks)

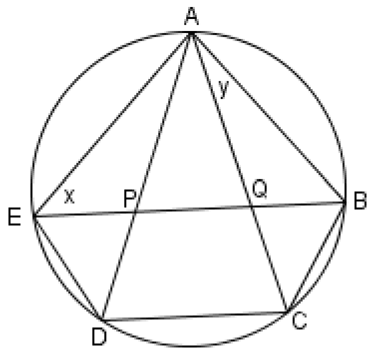
a) A chord of contact to the parabola $x^2 = 4y$, has the equation $y = x + 3$.
Determine the external point from which the tangents are drawn. 2

b) $ABCDE$ is a pentagon where $AB = AE$, $\angle AEB = x$ and $\angle BAC = y$.

i) Prove that $\angle AEB = \angle ECA$ 1

ii) Prove that $\angle BQC = x + y$ 2

iii) Prove that *PQCD* is a cyclic quadrilateral. 3



The End.

EXT 1 TASK 1 2014.

Q1

a) $t = \frac{x}{6}$

$y = 3\left(\frac{x}{6}\right)^2$

$= \frac{3x^2}{36}$

$= \frac{x^2}{12}$

$y' = \frac{x}{6}$

at $t = 2$ $x = 12$

$y' = 2$

$y - 12 = 2(x - 12)$

$y = 2x - 24 + 12$

$\boxed{y = 2x - 12}$

bii)

$x = ap^3 - yp + 2ap$

$ap^3 + yq + 2ap - yp = aq^3 + 2$

$y(q-p) = a(q^3 - p^3) + 2a(q-$

$y = a(q^2 + pq + p^2) + 2a$

$y = a(q^2 + pq + p^2 + 2)$

$\therefore$

$x = ap^3 - p(q^2 + pq + p^2 + 2)a +$

$= \cancel{ap^3} - pq^2 - p^2q - \cancel{ap^3} - \cancel{2ap} + 2$

$= -apq(p+q)$

Many students had poor setting out and didn't simplify their answer.

b) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

at $x = 2ap$

$m_1 = p$

$m_2 = -\frac{1}{p}$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

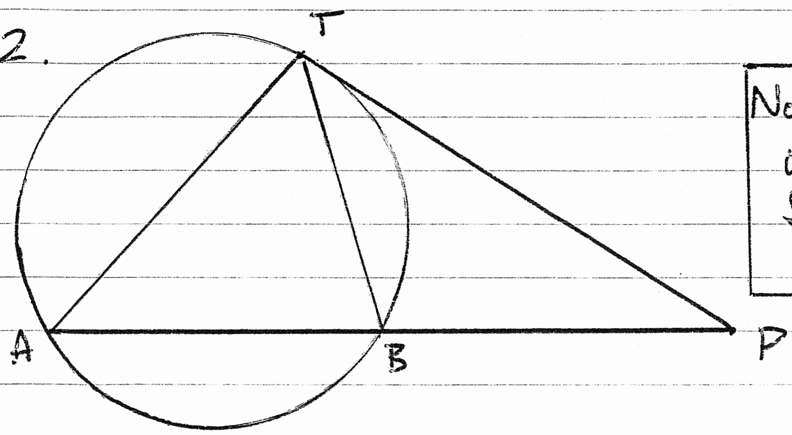
$yp - ap^3 = -x + 2ap$

$\boxed{x + yp = ap^3 + 2ap}$

$x + yq = aq^3 + 2aq$

Q2.

a.



Notes: Part a. done well. Another solution for ii used $AP.PB = PT^2$ and included angle P.

i. In $\triangle TBP$, $\triangle ATP$,

$\angle PTB = \angle PAT$ ($\angle$ between tangent and chord $= \angle$ in alternate segment)

$\angle BPT = \angle TPA$ ($\angle P$ is common)

$\therefore \triangle TBP \parallel\!| \triangle ATP$ (equiangular)

ii. $AP.PB = PT^2$ (square of tangent = product of intercepts of secant)

$\therefore 16 \times PB = 12^2$

$PB = \frac{144}{16} = 9\text{cm}$

b. Prove that $3^{4n} - 1 = 80p$ where p is an integer for $n \geq 1$

step 1: show true for $n=1$

$LHS = 3^4 - 1$
$= 81 - 1$
$= 80$
$= RHS$ with $p = 1$

step 2: assume true for $n=k$

ie, $3^{4k} - 1 = 80q$

step 3: prove true for $n = k+1$

ie, prove that: $3^{4(k+1)} - 1 = 80p$

$LHS = 3^{4k+4} - 1$
$= 3^{4k} \times 3^4 - 1$
$= (80q+1) \times 3^4 - 1$
$= 80q \times 3^4 + 3^4 - 1$
$= 80q \times 3^4 + 80$

$= 80(3^4 \times q + 1)$

$= 80p$ where $p = 3^4 \times q + 1$ and p is an integer.

$\therefore$ if the statement is true for $n=k$, then is also true for $n=k+1$.

$\therefore$ since true for $n=1$, also true for $n=1+1=2$, and hence for all integers $n \geqslant 1$, by the principle of Mathematical Induction.

Notes: a number of students tried to prove $3^{4k+1} - 1 = 80p$ instead of $3^{4(k+1)} - 1 = 80p$ and ran into difficulties. Be very careful with the initial algebra!

Q3 a)

i) $\angle COB = 70 \times 2$ ($\angle$ at the centre is twice
$= 140^\circ$ the $\angle$ at the circumf standing on the same Arc)

ii) $\angle OCB = \angle OBC$ (Base $\angle$'s of Isosceles Δ)

$2 \times \angle OCB = 180^\circ - 140^\circ$ (sum $\angle$'s of a Δ)

$\angle OCB = \frac{40}{2} = 20^\circ$

* All students did well in this question.

b) $4^n \geq 1 + 3n$ for $n \geq 1$

- Prove it is true for $n = 1$

$4^1 \geq 1 + 3$ which is true.

- Assume it is true for $n = k$

$4^k \geq 1 + 3k$ OR

$4^k - 3k - 1 \geq 0$ (*)

- Prove it is true for $n = k + 1$

$4^{k+1} - 3(k+1) - 1 \geq 0$

$4^{k+1} - 3k - 4 \geq 0$

$4(4^k - 3k - 4) + 9k \geq 0$

This is true as

$(4^k - 3k - 4) \geq 0$ from (*) and $9k \geq 0$

It is true for $n = k + 1$, but it is true for $n = 1$. It is proven by Mathematic for $n \geq 1$.

* There are several ways of proving this kind of question. Some students did not show the proper and logical prove.

2014 Ext 1 Task 1.

Q4 (a) $x = 2t-1$ $\quad y = 3t^2+1$

$t = \frac{x+1}{2}$

$y = 3\left(\frac{x+1}{2}\right)^2 + 1$

$y = 3\left(\frac{x^2+2x+1}{4}\right) + 1$

Comment: Students forgot to square the denominator. Also some didn't multiply the "+1" by 4.

$4y = 3x^2 + 6x + 7.$

or

$(x+1)^2 = \frac{4}{3}(y-1).$

(b)(i) $P(6p, 3p^2)$ $\quad Q(6q, 3q^2)$

$m_{PQ} = \frac{3p^2-3q^2}{6p-6q}$

$= \frac{1}{2}(p+q).$

Comment: Many students used the point $C(8,0)$ instead of P and Q.

eqn

$y - 3p^2 = \frac{1}{2}(p+q)(x-6p).$

$y - 3p^2 = \frac{1}{2}(p+q)x - 3p^2 - 3pq$

$y = \frac{1}{2}(p+q)x - 3pq$

(ii) $C(8,0)$

$0 = \frac{1}{2}(p+q)\times 8 - 3pq$

$3pq = 4(p+q).$

(iii)

$$M\left(\frac{6p+6q}{2}, \frac{3p^2+3q^2}{2}\right)$$

$$M\left(3(p+q), \frac{3}{2}(p^2+q^2)\right)$$

$$x = 3(p+q) \Rightarrow p+q = \frac{x}{3}.$$

$$y = \frac{3}{2}(p^2+q^2)$$

$$y = \frac{3}{2}\left((p+q)^2 - 2pq\right)$$

From part (ii) $pq = \frac{4}{3}(p+q)$

$$y = \frac{3}{2}\left((p+q)^2 - \frac{8}{3}(p+q)\right)$$

$$y = \frac{3}{2}\left(\left(\frac{x}{3}\right)^2 - \frac{8}{3}\left(\frac{x}{3}\right)\right)$$

$$= \frac{3}{2}\left(\frac{x^2}{9} - \frac{8x}{9}\right).$$

$$y = \frac{x^2}{6} - \frac{8x}{6}$$

$$6y = x^2 - 8x \quad \text{or} \quad (x-4)^2 = 4\left(\frac{3}{2}\right)\left(y+\frac{8}{3}\right)$$

5 a) i) $x^\circ = A\hat{B}P$ (alt $\angle$s, $AC \parallel BP$)

$A\hat{C}B = A\hat{B}P$ ($\angle$s in alt segments)

$\therefore x^\circ = A\hat{C}B$

$\therefore ABC$ is isosceles (base $\angle$s equal)

$\therefore AB = BC$

ii) $A\hat{C}B = B\hat{A}P$ ($\angle$s in alt segments)

$\therefore x^\circ = C\hat{A}B = B\hat{C}A = A\hat{B}P = B\hat{A}P$

$A\hat{B}C + 2x = 180^\circ$ ($\angle$ sum of $\triangle ABC$)

$A\hat{P}B + 2x = 180^\circ$ (" " " " ABP)

$\therefore A\hat{B}C = A\hat{P}B$

The angles are in the wrong position to use congruence.

b) $m_{OP} \times m_{OQ} = -1$

$$\frac{p^2 - 0}{2p - 0} \times \frac{q^2 - 0}{2q - 0} = -1$$

$$\frac{pq}{4} = -1$$

$\therefore pq = -4$

Finding the gradient of PQ does not help to answer the question

c) $E\hat{B}C + 95^\circ = 180^\circ$ (opp $\angle$s of a cycl quad)

$E\hat{B}C = 85^\circ$

$A\hat{F}E = E\hat{B}C$ (ext $\angle$ of a cyclic quad)

$= 85^\circ$

Question 6 (8 Marks)

a) Let $S(n): 1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots + n\times 2^n = (n-1)2^{n+1} + 2$

For $n=1$:

$LHS = n\times 2^n$ $\quad$ $RHS = (n-1)2^{n+1} + 2$

$= 1\times 2^1$ $\quad$ $= \cancel{(1-1)2^{1+1}} + 2$ ✓

$= 2$ $\quad$ $= 2$

$\therefore LHS = RHS$ $\quad \therefore S(n)$ is true for $n=1$.

Assume $S(k)$ is true for some integer k, $k \geq 1$

That is: $1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots + k\times 2^k = (k-1)2^{k+1} + 2$

Prove $S(k+1)$ is true. That is show that:

$1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots + k\times 2^k + (k+1)\times 2^{k+1} = (k+1-1)2^{k+1+1} + 2$ ✓

$= k.2^{k+2} + 2$

$LHS = \underbrace{1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots + k\times 2^k} + (k+1)2^{k+1}$

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$ ✓ (from assumption)

$= 2^{k+1}(k-1+k+1) + 2$

$= 2^{k+1} . 2k + 2$

$= k . 2^1 . 2^{k+1} + 2$

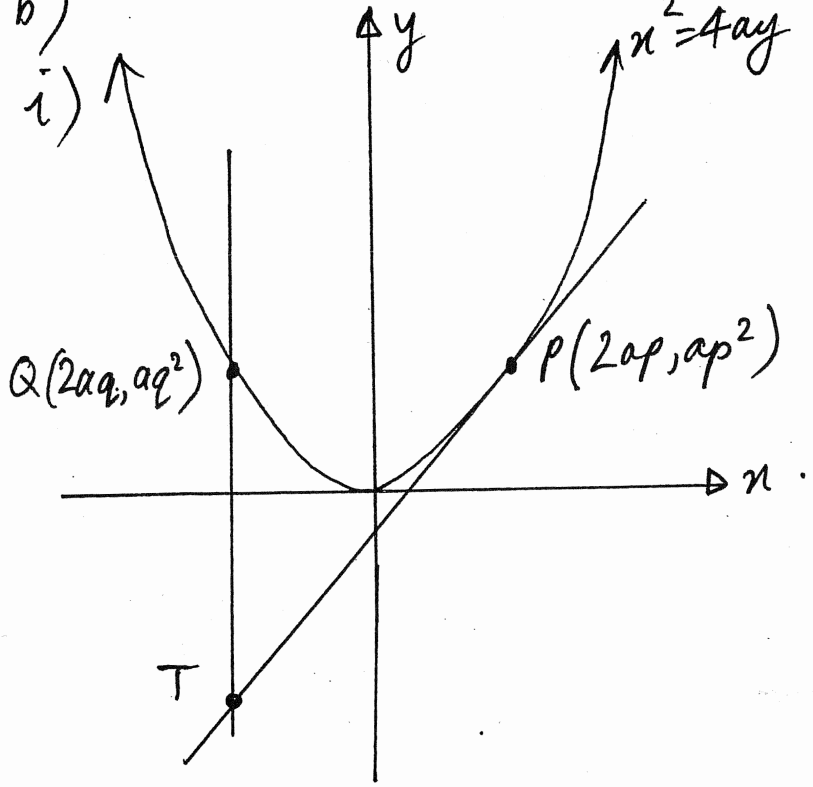
$= k . 2^{k+2} + 2$ ✓

$= RHS$

$\therefore S(n)$ is true for $n=k+1$, whenever it is true for $n=k$. $S(n)$ is true for $n=1$ and by the Principle of mathematical induction is true for all integers $n \geq 1$.

b)

i)



ii) when $x=2aq$: $y = p(2aq) - ap^2$
$= 2apq - ap^2$

$\therefore T = (2aq, 2apq - ap^2)$

iii) Midpoint of $PT = \left(\frac{2ap+2aq}{2}, \frac{2apq-\cancel{ap^2}+\cancel{ap^2}}{2} \right)$

$= \left(\frac{\cancel{2}a(p+q)}{\cancel{2}}, \frac{\cancel{2}apq}{\cancel{2}} \right)$

$= (a(p+q), apq)$.

Locus: $\begin{cases} x = a(p+q) \\ y = apq \end{cases}$

since $pq = -1$: $y = -a$

$\therefore$ Locus is the directrix of the parabola $x^2 = 4ay$.

Question 7

(a) (i) Let (x_1, y_1) be the external point.

$a = 1$

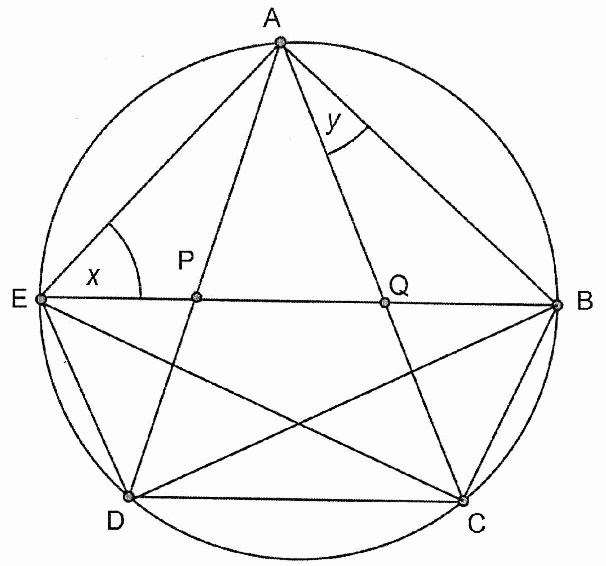
$xx_1 = 2a(y + y_1)$ $\quad\quad$ $\therefore xx_1 = 2(y + y_1)$

$y = \frac{x_1}{2}x - y_1$ $\quad\quad$ $\therefore \frac{x_1}{2} = 1, -y_1 = 3$

Hence the external point is $(2, -3)$.

COMMENT :

Many students did not recognise the opportunity to use the chord of contact formula and spent much longer on the solution than was required.



COMMENT :

Many students did not attempt this question fully – presumably due to time running out at the end of the exam period.

As with all proofs, please ensure reasoning is provided for each step of your explanation.

(b) (i) ΔBAE is isoceles (given, $AB = AE$)

$\angle ABE = \angle AEB = x$ (base $\angle$s of isos. Δ)

$\angle ABE = \angle ECA$ ($\angle$s in same segment)

$\therefore \angle AEB = \angle ECA = x$ (both equal $\angle ABE$)

(ii) $\angle ABE = x$ (proven above)

$\angle BQC = \angle ABQ + \angle BAQ$ (ext. $\angle$ of ΔAQB)

$\therefore \angle BQC = x + y$

(iii) $\angle BDC = \angle BAC = y$ ($\angle$s in same segment)

$\angle ADB = \angle AEB = x$ ($\angle$s in same segment)

$\therefore \angle PDC = \angle ADB + \angle BDC = x + y$

$\therefore PQCD$ is a cyclic quadrilateral

(ext. $\angle$ of quad. is equal to opp. int. $\angle$)